Appendix C: Notes on Plasticity Theory

Supplementary material for the course of Solid Mechanics

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C.1. Introduction

A typical uniaxial stress-strain curve is shown in Fig. C1. Here the nominal stress is used (load divided by original cross-sectional area) and the strain can be either the engineering strain, $\varepsilon = (L - L_0)/L_0$, where L is the current specimen length and L_0 the original length, or the logarithmic strain defined as $e = \ln(L/L_0)$. When the strains are small and higher order are neglected, either strain can be used since $e = \ln(L/L_0) = \ln(1+\varepsilon) = \varepsilon - \varepsilon^2/2 + \varepsilon^3/3 + ... \approx \varepsilon$.

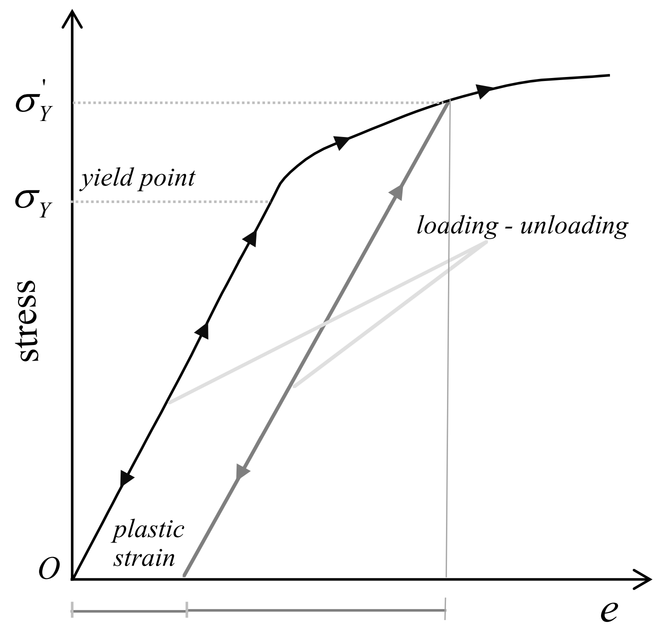

Fig. C1. Uniaxial stress - strain diagram for a strain-hardening material in tension with an initial yield stress σ_Y. Upon unloading after yielding, the path is parallel to the initial elastic phase. In reloading, the same path is followed until the new yield stress σ'_Y is reached.

Initially and up to the yield stress σ_Y, the material behaves in a linear elastic and reversible manner. In this phase of linear response, the theory of elasticity is adopted to model the constitutive response of the material. After the yield limit σ_Y, the material behaves in an inelastic manner with plastic deformation. From any point after yielding, unloading does not follow the same path demonstrating irreversible deformation. In a certain class of materials (especially metals), unloading and reloading is linear and parallel to the initial linear part as shown schematically in Fig. C1.

Upon loading in tension, various materials behave differently before and after yielding. Figure C2, shows schematics of stress-strain curves for different materials. These curves are approximations of real materials' response especially when the transition from elastic to plastic response is not smooth and are often used in stress analysis. When the transition is smooth as shown in Fig. 1C, a phenomenological relation known as *Ramberg–Osgood*, is established by fitting experimental data and used in computations.

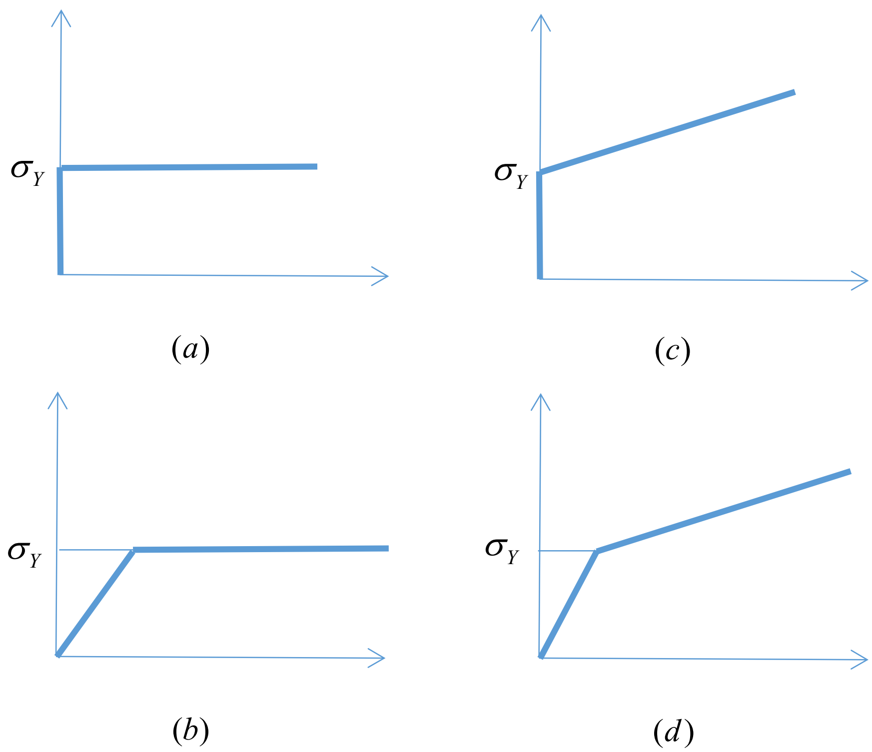

Fig. C2: Approximations of stress strain curves for: (a) rigid-perfectly plastic, (b) elastic-perfectly plastic, (c) rigid-linear hardening, (d) elastic-linear hardening material.

Modern approaches of stress analysis in the post-yield regime, are based on the incremental theories of plasticity. These theories aims at developing phenomenological constructive laws relating strains and stresses in nonlinear inelastic continuous media. With reference to a Cartesian coordinate system $x_i \quad (i=1,2,3)$, such constitutive relations are expressed in terms of increments of strain and stress tensors, $d\varepsilon_{ij}$ and $d\sigma_{ij}$, respectively. A basic assumption is that the strain tensor can be decomposed into elastic $d\varepsilon_{ij}^{e}$ and plastic $d\varepsilon_{ij}^{p}$ components (Fig. C1),

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p}. \qquad \text{(C.1)}$$

The elastic strain increment tensor is related to the stress increment tensor by means of Hooke's law of elasticity through the elastic compliance S_{ijkl} or stiffness C_{ijkl} tensor,

$$d\varepsilon_{ij}^{e} = S_{ijkl} d\sigma_{kl} \quad or \quad d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl}^{e}. \qquad \text{(C.2)}$$

Thus, the task of incremental plasticity theory lies in establishing the manner in which the plastic strain increment tensor is related to the stress field and history of deformation.

In this short summary, we review the basic concepts of plasticity theory and give some examples.

Yield criteria

The starting point in the theory is the criterion of yield in multiaxial loading conditions. The criteria leading to the choice of stress combination that will produce yielding are called *yield criteria*. Several such criterial have been proposed over the years. A number of them were put forward to predict strength of brittle materials and were later extended as yield criteria in ductile materials. A short description of the well-known yield criteria is given in the next section. The list is not exhaustive and other criteria can be found in the literature for isotropic and anisotropic materials.

Maximum stress theory. Also known as Rankine theory, it assumes that yielding occurs when one of the principal stresses in the structure becomes equal to the yield stress in simple tension. For example if σ_1 is the maximum principal stress and σ_3 is the minimum principal stress, yielding takes place when $\sigma_1 = \sigma_Y$ (tension) and (compression) $\sigma_3 = \sigma_{Y,c}$. When yielding in tension and compression are assumed equal, the criterion becomes,

$$\sigma_1 = \sigma_Y \text{ or } \sigma_3 = -\sigma_Y. \qquad \text{(C.3)}$$

The theory is not in good agreement with experimental data and is rarely used.

Maximum Strain Theory. In this theory, also called *Saint-Venant Theory*, yielding occurs when the maximum value of the principal strain equals the values of the yield strain in simple tension or compression. For example if ε_1 is the largest strain in absolute value, yielding occurs when,

$$E\varepsilon_1 = \sigma_1 - \nu(\sigma_2 + \sigma_3) = \pm\sigma_Y. \qquad \text{(C.4)}$$

This theory also is in poor agreement with experimental data.

Maximum Strain Energy Theory, or Beltrami's Energy Theory. Here it is assumed that yielding occurs when the total strain energy per unit volume at yield equals the strain energy per unit volume at yield in uniaxial tension or compression. In a tensile test, the stain energy per unit volume at yield is,

$$W = \frac{1}{2}\sigma_Y \varepsilon_Y = \frac{1}{2E}\sigma_Y^2. \qquad \text{(C.5a)}$$

In a general loading case, the total strain energy per unit volume is,

$$W = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] = \frac{1}{2E}\sigma_Y^2. \qquad \text{(C.5b)}$$

Thus, the yield criterion becomes,

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_Y^2. \qquad \text{(C.5c)}$$

The main drawback of this theory is that it predicts yielding under relatively high pressures which is not observed experimentally.

Maximum Shear Theory, or Tresca Criterion. The theory, also called Coulomb Theory, assumes that yielding occurs when the maximum shear stress becomes equal to the value of maximum shear stress in simple tension. We know that in simple tension, the maximum shear is $\frac{1}{2}\sigma_Y$ and if the principal stresses are known $\sigma_1 > \sigma_2 > \sigma_3$, the criterion is expressed as,

$$\frac{1}{2}(\sigma_1 - \sigma_3) = \pm\frac{1}{2}\sigma_Y. \qquad \text{(C.6a)}$$

Note that in simple tension, we have $\sigma_1 = \sigma_Y$.This Criterion is in satisfactory agreement with experimental data and is used often due to its simplicity, when the principal stresses are known.

Mohr-Coulomb Criterion. In this theory, it is assumed that while the maximum shear stress is the cause of yielding (Tresca Criterion), its action is reduced by a normal stress acting on the plane of the shear stress. Further, the theory considers a linear relationship between the shear stress and normal stress on the failure plane as an effort to account for internal friction and explain yielding in certain materials. As in the case of Tresca Criterion, the intermediate principal stress σ_2 does not play any role. Thus, yielding occurs when the following condition is met,

$$\sigma_1 - \frac{\sigma_Y}{\sigma_{Y,c}}\sigma_3 = \sigma_Y \tag{C.6b}$$

where $\sigma_{Y,c}$ is the yield stress in compression. When this stress is equal to the yield stress in tension $\sigma_{Y,c} = \sigma_Y$, the criterion reduces to Tresca Criterion defined earlier. This theory is applied to geomaterials where some frictional effects appear in yielding.

Distortion Energy Criterion. This theory also known as *Von Mises Yield Criterion*, assumes that yielding starts when the distortion energy becomes equal to the distortion energy at yield in simple tension. It is based on experimental evidence that the volumetric component of the strain energy (or hydrostatic stress) does not contribute to plastic deformation.

Energy analysis shows that the distortion energy per unit volume is given by,

$$W_d = \frac{1}{2G} I_2(\boldsymbol{s}) \tag{C.7a}$$

where $G = \mu$ is the shear modulus and $I_2(\boldsymbol{s})$ the second invariant of the deviatoric stress tensor which in terms of the principal stresses is,

$$I_2(\boldsymbol{s}) = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]. \tag{C.7b}$$

In uniaxial tension at yield we have,

$$I_2(\boldsymbol{s}) = \frac{1}{3}\sigma_Y^2. \tag{C.7c}$$

Thus, the criterion based on distortion energy becomes,

$$\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = 2\sigma_Y^2 \ . \tag{C.7d}$$

In terms of the 6 components of stress tensor it is,

$$\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)\right] = 2\sigma_Y^2. \tag{C.7e}$$

Note that, this criterion can be stated in terms of $I_2(s)$: yielding occurs when the second invariant of the deviatoric stress tensor becomes equal to the corresponding one in uniaxial tension. It gives satisfactory results and is often used due its simplicity. In simple tension we have $\sigma_{11} = \sigma_1 = \sigma_Y$.

The V. Mises criterion is also stated in terms of the octahedral stress because of the relation,

$$\begin{aligned} \tau_{oct}^2 &= \frac{2}{3} I_2(s) = \frac{1}{9}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \\ \tau_{oct} &= \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}. \end{aligned} \tag{C.7f}$$

Thus, yielding takes place when the octahedral shear stress becomes equal to that in uniaxial tension, given by,

$$\tau_{oct} = \frac{\sqrt{2}}{3} \sigma_Y . \tag{C.7g}$$

In a biaxial stress state, relation (C.7d) reduces to,

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = \sigma_Y^2 . \tag{C.7h}$$

which is an ellipse in the $\sigma_1 - \sigma_2$ plane as shown in Fig. C3a.

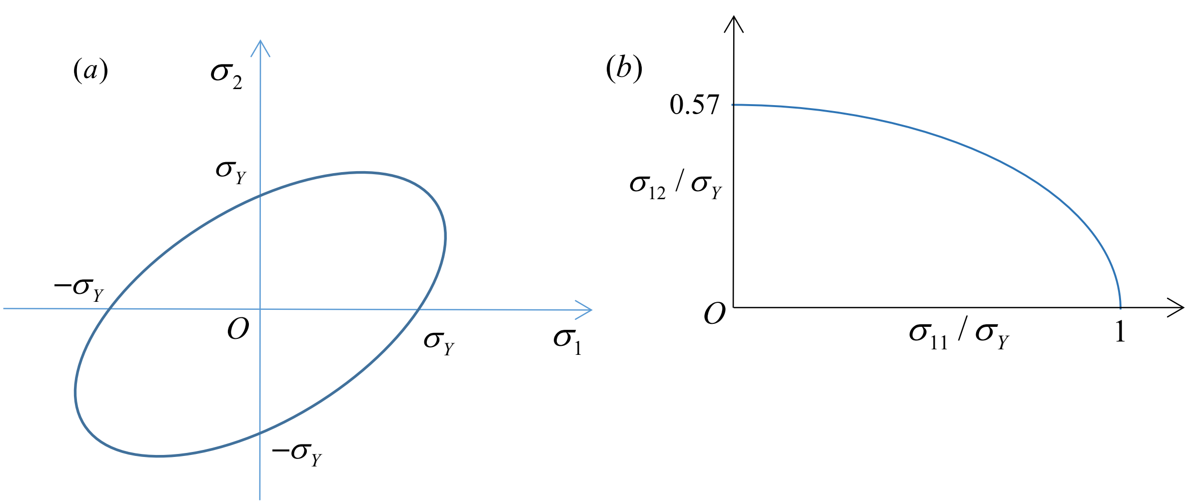

Fig. C3: V. Mises ellipse in a biaxial stress state with equal yielding in tension and compression. (b) The same criterion in traction torsion loading.

Example 1. An important experiment to carry out multiaxial stress testing in the laboratory is a thin tube that is simultaneously subjected to traction, torsion and internal pressure. Suppose that this experiment is performed only under tension and torsion. Express the V. Mises yield condition for this experiment.

Solution

The stress state at a point in the tube is σ_{11} due to traction and σ_{12} due to torsion[1]. Thus, (C.7e) reduces to,

$$\left(2\sigma_{11}^2 + 6\sigma_{12}^2\right) = 2\sigma_Y^2 \quad \Rightarrow \quad \left(\frac{\sigma_{11}}{\sigma_Y}\right)^2 + 3\left(\frac{\sigma_{12}}{\sigma_Y}\right)^2 = 1 .$$

The last equation is an ellipse as shown in Fig. C3b. When the stress state is represented by a point within the ellipse, the materials is in the linear elastic range. A point in the ellipse signifies yielding and any stresses state outside the ellipse is not permitted.

Yield Surface[2]

It is clear from the discussion on the preceding section that yield criteria are expressed in terms of stresses. For biaxial stress state problems, these criteria are presented as curves[3] (Fig. C3).

In the most general case, the yield criteria will depend on the stress state at a point as given by the symmetric Cauchy stress tensor. It is assumed therefore, that for a pristine material under multiaxial stresses, the yield criterion is an extension of yielding in uniaxial loading and the criterion is expressed as,

$$f(\sigma_{ij}) = K . \tag{C.8}$$

Here f is a so-called the *yield function* and K is a known parameter that reflects the material. In 6-dimensional stress space, relation (C.8) is represented by the yield surface. Any point on that surface corresponds to a stress state at which yielding can begin. A point inside the surface corresponds to a state in the linear domain and a point outside the surface is not permitted.

To continue, we adopt two hypotheses:

1. For an isotropic material, the yield function is represented only in terms of the three principal

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[1] The stresses can be expressed in terms of the applied torque and axial load using well known approximations form structural mechanics.
[2] In the next paragraphs of this summary, the plasticity theory based on the V. Misses yield criterion is discussed.
[3] They can also be a polygon, i.e. the Tresca criterion.

stresses, or the three invariants of the stress tensor.

2. Based on experimental evidence on several materials, the hydrostatic stress does not contribute to yielding. Thus, (C.8) is expressed only in terms of the invariants of the deviatoric stress tensor ***s***,

$$f_1\left(I_1(\boldsymbol{s}), I_2(\boldsymbol{s}), I_3(\boldsymbol{s})\right) = K \tag{C.9}$$

where $I_i(s_i)$ $(i = 1,2,3)$ are the invariants of the deviatoric stress tensor ***s***. Alternatively, the yield function can be expressed in terms of the principal deviatoric stress components s_i $(i = 1,2,3)$,

$$\begin{aligned} I_1(\boldsymbol{s}) &= s_1 + s_2 + s_3 = 0 \\ I_2(\boldsymbol{s}) &= \frac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right) \\ I_3(\boldsymbol{s}) &= s_1 s_2 s_3 . \end{aligned} \tag{C.10}$$

or,

$$f_2\left(I_2(\boldsymbol{s}), I_3(\boldsymbol{s})\right) = K . \tag{C.11}$$

The well-known V. Mises yield criterion is a specific case of the latter equation,

$$f_3(I_2(s)) = I_2(s) = J_2 = \tfrac{1}{3}\sigma_Y^2 \quad . \tag{C.12}$$

<u>Note that for convenience in the following, the second invariant of the deviatoric stress tensor will be indicated by</u> J_2.

It is useful to study the yield surface in space. Thus, we introduce a coordinate system whose axes are the three principal stresses $\sigma_1, \sigma_2, \sigma_3$ that defines a stress space, called *Westergard Stress Space*. Every point in this space represents a possible stress state.

Consider next the straight line *ON* having equal angles with the three axes, i.e. $\cos(ON,\sigma_1) = \cos(ON,\sigma_2) = \cos(ON,\sigma_3) = 1/\sqrt{3}$ (Fig. C4). For any point on this line $\sigma_1 = \sigma_2 = \sigma_3$ and $s_1 = s_2 = s_3 = 0$.

Accordingly, any point on this line represents a hydrostatic stress state. Consider next, a plane normal to *ON* whose equation is,

$$\sigma_1 + \sigma_2 + \sigma_3 = \sqrt{3}\rho . \tag{C.13}$$

Here ρ is the distance of the plane from the origin. Note that along this line, the hydrostatic stress increases linearly with distance from the origin. The plane, with $\rho = 0$, passes by the origin and is called the π *plane*. We will show now that in the stress space, the V. Mises criterion (C.12) is represented by an open-ended cylinder.

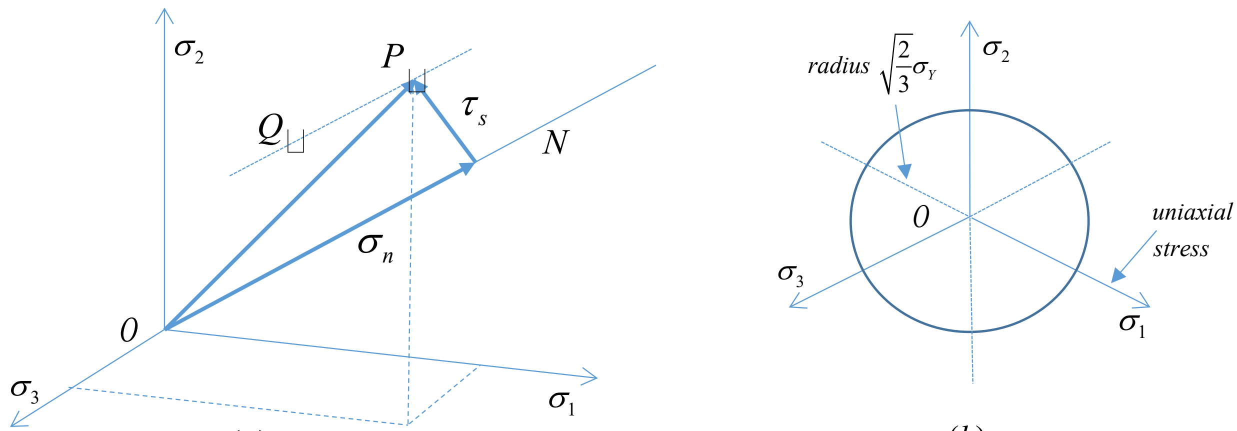


Fig. C4: (a) Westergard Stress Space. ON makes the same angle with the axes. P represents a stress state. A stress state at Q on a line parallel to ON has the same t_s. *(b) V. Mises criterion on* π *plane*.

Consider a stress state $\sigma_1, \sigma_2, \sigma_3$ represented by a point *P* (Fig. C4). The components of stress vector *OP* parallel and normal to *ON* are,

$$\sigma_n = \frac{1}{\sqrt{3}}\sigma_1 + \frac{1}{\sqrt{3}}\sigma_2 + \frac{1}{\sqrt{3}}\sigma_3 = \frac{1}{\sqrt{3}}\left(\sigma_1 + \sigma_2 + \sigma_3\right) \tag{C.14}$$

$$\tau_s^2 = \boldsymbol{OP}^2 - \sigma_n^2 = \sigma_1^2 + \sigma_2^2 + \sigma_2^2 - \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2 = 2J_2\,. \tag{C.15}$$

The shear component can be further expressed as follows,

$$\tau_s^2 = s_1^2 + s_2^2 + s_2^2 = 2J_2\,. \tag{C.16}$$

Note that the components of the shear stress component on the π *plane* are the components of the deviatoric stress tensor. It is not difficult to see that for another stress state, represented by point *Q*, on the line parallel to *ON*, the shear stress component is equal to that of point *P* and the normal component varies according to (C.13). Thus, we can conclude that, for any point on cylinder whose axis is *ON* and radius given by $\tau_s = \sqrt{2J_2}$, the shear stress is the same. The circle shown in Fig. C4b, is the intersection of the cylinder with π *plane*. For the V. Mises criterion (C.12),

$$J_2 = \frac{1}{3}\sigma_Y^2 \tag{C.17}$$

and (C.16), the radius of this circle is (Fig. C4b),

$$r = \sqrt{\frac{2}{3}}\sigma_Y \,. \tag{C.18}$$

Subsequent yield surfaces

Upon loading, a virgin material will yield when the yield criterion is satisfied. For a perfectly plastic material (Fig. C2a, b), the yield stress and yield surface remain the same upon further loading. In several material, however, the stress-strain curve rises and thus, the yield stress increases upon further loading. This phenomenon is called *strain-hardening,* or *work hardening* (Fig. 1C). As a result the yield surface changes upon loading beyond σ_Y. To describe this phenomenon we need to define a yield function for general loading,

$$f\left(\sigma_{ij}\right) = K \,. \tag{C.8bis}$$

When the equality is met, yielding begins and in the stress space we have the initial yield surface. For a strain-hardening material, parameter *K* takes on a new value and as in the case of the yield stress in uniaxial test, if the material is unloaded and reloaded, additional yielding does not occur until the new value of *K* is reached (in the uniaxial test the yield stress increases from σ_{Y} to σ'_{Y}, see Fig. 1C). The foregoing concepts are formalized as follows,

1. Loading

$$f\left(\sigma_{ij}\right) = K, \quad \frac{\partial f\left(\sigma_{ij}\right)}{\partial \sigma_{ij}} d\sigma_{ij} > 0 \tag{C.19}$$

2. Neutral Loading

$$f\left(\sigma_{ij}\right) = K, \quad \frac{\partial f\left(\sigma_{ij}\right)}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \tag{C.20}$$

3. Unloading

$$f\left(\sigma_{ij}\right) = K, \quad \frac{\partial f\left(\sigma_{ij}\right)}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \tag{C.21}$$

These three conditions imply that in loading, the stress increment tends outwards from the yield surface producing plastic deformation. In neutral loading, the stress increment points on a tangential direction to the surface and in unloading it points inwards.

At this point, it is important to define *isotropic hardening* and the *Bauschinger effect*. The stress-strain curve for an isotopic material with identical behaviors in in tension and compression is shown in Fig. C5a.

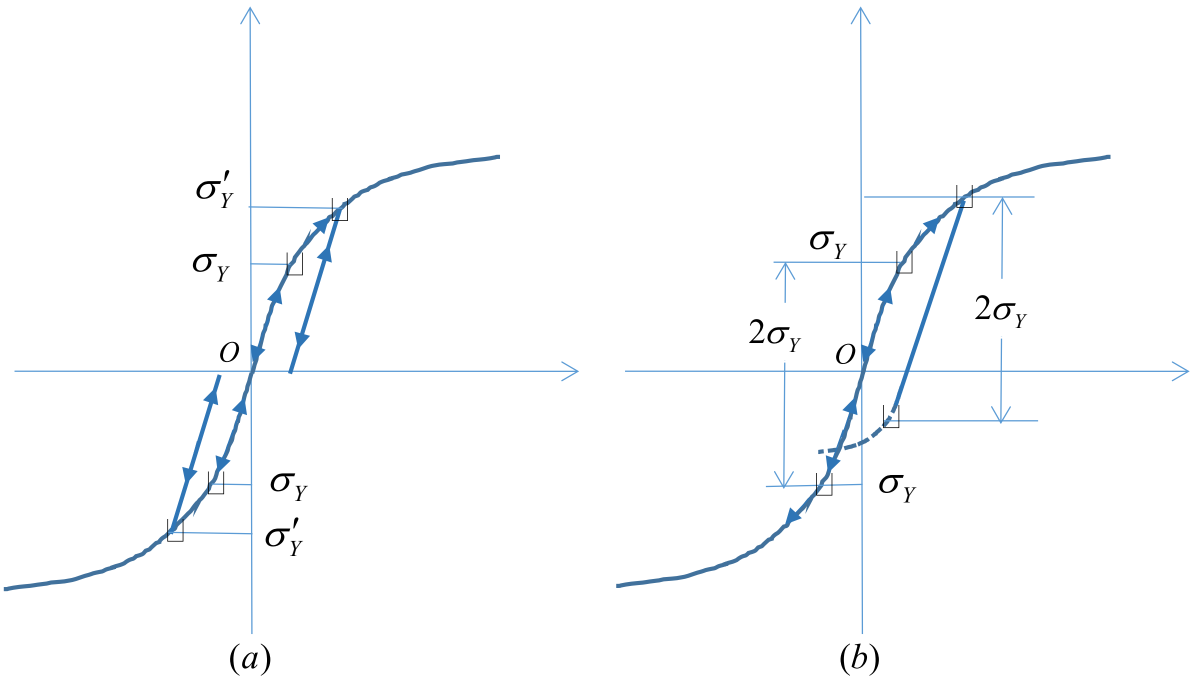
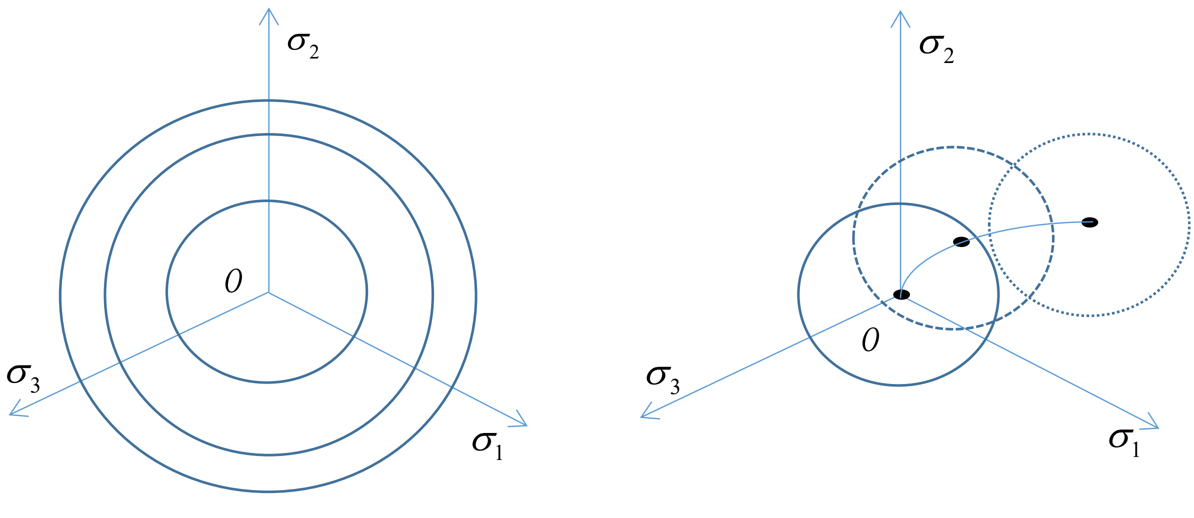


Fig. C5: (a) tension compression curve of a material with the same tension and compression defining isotropic hardening. (b) Definition of kinematic hardening. The yield stresses change in tension and compression but the elastic stress 'distance' between tension and compression remains the same.

We notice here that the behavior (post-yield response) is identical in tension and compression. During repeated loading and unloading the yield stress increases, due to strain hardening, by the same amount in tension and compression.

This is a so-called *isotropic hardening* material and for the V. Mises yield criterion, the initial cylinder in the stress space expands with hardening while maintaining its shape. In the π *plane*, these cylindrical surfaces are concentric circles as shown in Fig. C6. Isotropic hardening is often adopted as good first order approximation to model plasticity in certain materials because of its simplicity.

However, experimental results have shown that isotropic hardening is not always satisfied and the material behavior in compression is not identical to that in tension[4]. Thus, the yield surfaces do not only expand but also change shape during yielding.



(a) (b)

Fig. C6: Schematics of (a) Yield loci of the V. Mises yield surface on the π plane for a strain hardening material. (b) Yield loci for a kinematic hardening.

This phenomenon is attributed to the so-called *Bauschinger* effect. This effect reduces the yield stress in compression if the material is strain hardened in tension, unloaded and reloaded in compression as shown schematically in Fig C5b. To account for the *Bauschinger* effect, Prager has introduced a simplified model called *kinematic hardening*. In this model the total elastic range is maintained constant during loading and the surface translates undeformed in the stress space. In most real materials, strain- and kinematic hardening, are manifested and the yield surface changes its shape and moves in the stress space.

Plastic strains

After yielding, the material deforms in an elastoplastic manner and elastic and plastic strains are produced. There are two types of theories to model plastic strains. The first one encompass the so-called *incremental* or *flow theories of plasticity* and relate plastic strain increments to current stress level. Such theories are based on experimental results showing that plastic strains depend on the loading path. Thus, the increments of strains are computed throughout the loading history and expressed in terms of the current stress level. To determine the total plastic strains, we integrate

[4] The material microstructure changes due to plastic strains in tension that lowers the compressive yield stress.

the incremental stress-strain relations over the entire history of loading.

The second type encompass what are called *total* or *deformation theories of plasticity.* Here the total strain components are related to the current stress. In the following sections, we review a well-known incremental plasticity theory. A brief description of a total deformation theory is given at the end of this Appendix.

Prandtl-Reuss equations[5]

It was stated earlier that the total strain tensor can be decomposed in elastic and plastic components (relation C.1). While the elastic part follows Hook's law (C.2), the plastic part is expressed by the Prandtl-Reuss equations,

$$\frac{d\varepsilon_{11}^p}{s_{11}}=\frac{d\varepsilon_{22}^p}{s_{22}}=\frac{d\varepsilon_{33}^p}{s_{33}}=\frac{d\varepsilon_{12}^p}{s_{12}}=\frac{d\varepsilon_{23}^p}{s_{23}}=\frac{d\varepsilon_{31}^p}{s_{31}}=d\lambda\geq 0 \qquad \text{(C.22a)}$$

In index form they are,

$$d\varepsilon_{ij}^p=s_{ij}d\lambda \qquad \text{(C.22b)}$$

Here $d\varepsilon_{ij}^p$ are the plastic strain increments, s_{ij} the components of the deviatoric stress tensor and $d\lambda$ is an non negative parameter that may change with loading. Note also that (C.22) imply,

$$d\varepsilon_{11}^p+d\varepsilon_{22}^p+d\varepsilon_{33}^p=0 \qquad \text{(C.23)}$$

Equations (C.22) state that $d\varepsilon_{ij}^p$ are proportional to s_{ij} and not to the stress increment. In terms of the principal plastic strain increments and principal stresses, relations (C.22a) simplify to,

$$\frac{d\varepsilon_1^p}{s_1}=\frac{d\varepsilon_2^p}{s_2}=\frac{d\varepsilon_3^p}{s_3}=d\lambda \qquad \text{(C.24a)}$$

which easily leads to,

$$\frac{d\varepsilon_1^p-d\varepsilon_2^p}{s_1-s_2}=\frac{d\varepsilon_2^p-d\varepsilon_3^p}{s_2-s_3}=\frac{d\varepsilon_3^p-d\varepsilon_1^p}{s_3-s_1}=d\lambda \qquad \text{(C.24b)}$$

In terms of the stress components, (C22) are,

[5] The Prandtl-Reuss theory is one of the earlier one to model plastic strains.

$$d\varepsilon_{11}^{p} = d\lambda s_{11} = \frac{2}{3} d\lambda \left[\sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33}) \right]$$

$$\ldots\ldots\ldots \qquad \text{(C.25)}$$

$$d\varepsilon_{12}^{p} = d\lambda s_{12} = d\lambda \sigma_{12}$$

$$\ldots\ldots\ldots$$

Thus, if $d\lambda$ is known, the plastic strain increments can be calculated using (C.22) or (C.25). In plasticity, relation (C.24) or (C.25) is known as *flow rule*[6].

With the flow rule known, the full elastic-plastic stress- strain relations are,

$$d\varepsilon_{11} = d\varepsilon_{11}^{e} + d\varepsilon_{11}^{p} = \frac{1}{E}\left[d\sigma_{11} - v(d\sigma_{22} + d\sigma_{33})\right] + \frac{2}{3} d\lambda \left[\sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33}) \right]$$

$$\ldots\ldots\ldots \qquad \text{(C.26a)}$$

$$d\varepsilon_{12} = d\varepsilon_{12}^{e} + d\varepsilon_{12}^{p} = \frac{1+v}{E} d\sigma_{12} + d\lambda \sigma_{12}$$

$$\ldots\ldots\ldots$$

or in index form,

$$d\varepsilon_{ij} = \frac{1+v}{E} d\sigma_{ij} - \frac{v}{E}\delta_{ij} d\sigma_{kk} + d\lambda s_{ij} \qquad \text{(C.26b)}$$

In summary, (C.26) are called that *Prandtl-Reuss* equations. When the elastic part is neglected, the equations are known as the *Lévy-Mises* equations.

Example 2: (1) Express the *Prandtl-Reuss* equations in terms of the principal stresses, (2) Show that the principal axes of the plastic strain increments and principal stresses coincide.

Solution

1. From (C.24a) we write,

$$d\varepsilon_{1}^{p} = s_1 d\lambda = \left(\sigma_1 - \frac{1}{3}\sigma_{ii} \right) d\lambda; \quad d\varepsilon_{2}^{p} = s_2 d\lambda = \left(\sigma_2 - \frac{1}{3}\sigma_{ii} \right) d\lambda;$$

$$d\varepsilon_{3}^{p} = s_3 d\lambda = \left(\sigma_3 - \frac{1}{3}\sigma_{ii} \right) d\lambda \qquad \text{(a)}$$

[6] Other flow rules are found in the literature with various degrees of sophistication.

2. Expressing $s_1 - s_2 = \left(\sigma_1 - \frac{1}{3}\sigma_{ii}\right) - \left(\sigma_2 - \frac{1}{3}\sigma_{ii}\right) = \sigma_1 - \sigma_2, \ldots$ (b)

it is easy to see that (C.24b) become,

$$\frac{d\varepsilon_1^p - d\varepsilon_2^p}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2^p - d\varepsilon_3^p}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3^p - d\varepsilon_1^p}{\sigma_3 - \sigma_1} = d\lambda \qquad \text{(c)}$$

The last relation shows that the principal directions coincide.

To calculate the strains, parameter $d\lambda$ needs to be specified and for this a yield criterion is required. This is shown in the next paragraphs. We start with (C.22) and replace the deviatoric components s_{ij} with their expressions in terms of σ_{ij}. Subsequently we can show that,

$$\begin{aligned}&\left(d\varepsilon_{11}^p - d\varepsilon_{22}^p\right)^2 + \left(d\varepsilon_{22}^p - d\varepsilon_{33}^p\right)^2 + \left(d\varepsilon_{33}^p - d\varepsilon_{11}^p\right)^2 + 6\left(d\varepsilon_{12}^p\right)^2 + 6\left(d\varepsilon_{23}^p\right)^2 + 6\left(d\varepsilon_{31}^p\right)^2 \\ &= \left(d\lambda\right)^2\left[\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2 + 6\left(\sigma_{12}\right)^2 + 6\left(\sigma_{23}\right)^2 + 6\left(\sigma_{31}\right)^2\right]\end{aligned} \qquad \text{(C.27)}$$

Note that the expression in the brackets of the right hand is equal to $9\tau_{oct}^2$. We then define the following parameter, called octahedral strain,

$$\left(d\gamma_0^p\right)^2 = \frac{1}{9}\left[\left(d\varepsilon_{11}^p - d\varepsilon_{22}^p\right)^2 + \left(d\varepsilon_{22}^p - d\varepsilon_{33}^p\right)^2 + \left(d\varepsilon_{33}^p - d\varepsilon_{11}^p\right)^2 + 6\left(d\varepsilon_{12}^p\right)^2 + 6\left(d\varepsilon_{23}^p\right)^2 + 6\left(d\varepsilon_{31}^p\right)^2\right].$$

(C.28)

Using the above expressions and relation between octahedral stress τ_{oct} and second invariant of the deviatoric stress tensor J_2, $\tau_{oct}^2 = \frac{2}{3}J_2$, we obtain,

$$d\lambda = \frac{d\gamma_0^p}{\tau_{oct}} = \sqrt{\frac{3}{2}}\frac{d\gamma_0^p}{\sqrt{J_2}}. \qquad \text{(C.29)}$$

To develop further this model, we define an equivalent stress σ_e and equivalent plastic strain increment $d\varepsilon_p$ as follows,

$$\sigma_e = \frac{1}{\sqrt{2}}\left[\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2 + 6\left(\sigma_{12}\right)^2 + 6\left(\sigma_{23}\right)^2 + 6\left(\sigma_{31}\right)^2\right]^{1/2}$$

$$= \sqrt{3J_2} = \frac{3}{\sqrt{2}}\tau_{oct} \tag{C.30}$$

$$d\varepsilon_p = \frac{\sqrt{2}}{3}\left[\left(d\varepsilon_{11}^p - d\varepsilon_{22}^p\right)^2 + \left(d\varepsilon_{22}^p - d\varepsilon_{33}^p\right)^2 + \left(d\varepsilon_{33}^p - d\varepsilon_{11}^p\right)^2 + 6\left(d\varepsilon_{12}^p\right)^2 + 6\left(d\varepsilon_{23}^p\right)^2 + 6\left(d\varepsilon_{31}^p\right)^2\right]^{1/2}$$

$$= \sqrt{2}d\gamma_0^p \tag{C.31}$$

Accordingly, $d\lambda$ is expressed as,

$$d\lambda = \frac{3}{2}\frac{d\varepsilon_p}{\sigma_e}. \tag{C.32}$$

and relations (C.25) become,

$$d\varepsilon_{11}^p = d\lambda s_{11} = \frac{d\varepsilon_p}{\sigma_e}\left[\sigma_{11} - \frac{1}{2}(\sigma_{22} + \sigma_{33})\right]$$

....

$$d\varepsilon_{12}^p = \frac{3}{2}\frac{d\varepsilon_p}{\sigma_e}s_{12} \tag{C.33a}$$

....

or in index form,

$$d\varepsilon_{ij}^p = \frac{3}{2}\frac{d\varepsilon_p}{\sigma_e}s_{ij} \tag{C.33b}$$

It is interesting to notice here that the equivalent stress σ_e in (C.30) gives the V. Mises criterion and according to (C.6d) yielding begins when,

$$\sigma_e = \sigma_Y \tag{C.34}$$

Thus, σ_e is the V. Mises yield function and because it is used in (C.33), the Prandtl-Reuss relations imply the V. Mises yield criterion.

Example 3: Show that the equivalent stress (C.30) can be written as,

$$\sigma_e = \left[\frac{3}{2}s_{ij}s_{ij}\right]^{1/2} = \left[\frac{3}{2}(s_1^2 + s_2^2 + s_3^2)\right]^{1/2} \tag{a}$$

Solution

Express the deviatoric stress s_{ij} in terms of σ_{ij} and carry out the multiplications,

$$\sigma_e^2 = \frac{3}{2}s_{ij}s_{ij} = \frac{3}{2}\left(\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{mm}\right)\left(\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}\right)$$
$$= \frac{3}{2}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{ij}\sigma_{kk} - \frac{1}{3}\delta_{ij}\sigma_{ij}\sigma_{mm} + \frac{1}{9}\delta_{ij}\delta_{ij}\sigma_{kk}\sigma_{mm}\right)$$
$$= \frac{3}{2}\left(\sigma_{ij}\sigma_{ij} - \frac{2}{3}\sigma_{ll}\sigma_{kk} + \frac{1}{3}\sigma_{kk}\sigma_{mm}\right) = \frac{3}{2}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}\left(\sigma_{kk}\right)^2\right)$$

$$\sigma_{ij}\sigma_{ij} = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{31}^2$$
$$\left(\sigma_{kk}\right)^2 = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{11}\sigma_{22} + 2\sigma_{22}\sigma_{33} + 2\sigma_{33}\sigma_{11}$$

Thus,

$$\sigma_e^2 = \frac{3}{2}\left[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{31}^2 - \frac{1}{3}(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{11}\sigma_{22} + 2\sigma_{22}\sigma_{33} + 2\sigma_{33}\sigma_{11})\right]$$
$$\sigma_e^2 = \frac{3}{2}\frac{1}{3}\left[3\left(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{31}^2\right) - (\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{11}\sigma_{22} + 2\sigma_{22}\sigma_{33} + 2\sigma_{33}\sigma_{11})\right]$$
$$\sigma_e^2 = \frac{1}{2}\left[\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2 + 6\sigma_{12}^2 + 6\sigma_{23}^2 + 6\sigma_{31}^2\right].$$

The right hand side of (a) is expressed in terms of the principal values and obtained by expanding $s_{ij}s_{ij}$ and noticing that the shear components are zero,

$$\sigma_e = \left[\frac{3}{2}\left(s_{11}s_{11} + s_{22}s_{22} + s_{33}s_{33}\right)\right]^{1/2} = \left[\frac{3}{2}(s_1^2 + s_2^2 + s_3^2)\right]^{1/2}.$$

Example 4: Show that the equivalent plastic strain increment (C.31) can be written as,

$$d\varepsilon_p = \left[\frac{2}{3}d\varepsilon_{ij}^p d\varepsilon_{ij}^p\right]^{1/2} = \left[\frac{2}{3}\left(\left(d\varepsilon_1^p\right)^2 + \left(d\varepsilon_2^p\right)^2 + \left(d\varepsilon_3^p\right)^2\right)\right]^{1/2}.$$

Solution

We follow the same steps as in the previous example by replacing the plastic strain increments and carrying out the calculations, i.e.,

$$d\varepsilon_p = \left[\frac{2}{3}d\varepsilon_{ij}^p d\varepsilon_{ij}^p\right]^{1/2} = \left[\frac{2}{3}\left(d\varepsilon_{ij}^p - \frac{1}{3}\delta_{ij}d\varepsilon_{mm}^p\right)\left(d\varepsilon_{ij}^p - \frac{1}{3}\delta_{ij}d\varepsilon_{kk}^p\right)\right]^{1/2}.$$

Since the structure of this relation is identical to the first relation in the previous Example (except the coefficient) we follow the same steps. Also, it is interesting to notice that we can write,

$$d\varepsilon_p = \frac{\sqrt{2}}{3}\Big[\left(d\varepsilon_{11}^p - d\varepsilon_{22}^p\right)^2 + \left(d\varepsilon_{22}^p - d\varepsilon_{33}^p\right)^2 + \left(d\varepsilon_{33}^p - d\varepsilon_{11}^p\right)^2 + 6\left(d\varepsilon_{12}^p\right)^2 + 6\left(d\varepsilon_{23}^p\right)^2 + 6\left(d\varepsilon_{31}^p\right)^2\Big]^{1/2}$$

which is relation (C.31).

We can distinguish now the two important cases in plasticity, according to (C.33b):

Perfectly plastic material. Here σ_Y does not change (Fig. C2a,b) and thus, we have,

$$d\varepsilon_{ij}^p = \frac{3}{2}\frac{d\varepsilon_p}{\sigma_Y}s_{ij} \,. \tag{C.34}$$

Work hardening or Strain hardening material. When the material work-hardens, or strain-hardens (Fig. C1), the yield stress increases, $\sigma_e > \sigma_Y$. In such cases, it is necessary to establish a relation between $d\varepsilon_p$ and σ_e in (C.33). In this plasticity model, such relation is expresses in terms of the total plastic work, or equivalent plastic strains. Below, we state these relationships without going into the details.

Work hardening: The work increment per unit volume during deformations is,

$$dW = \sigma_{ij} d\varepsilon_{ij} \tag{C.35a}$$

Using (C.1) we can write,

$$dW = \sigma_{ij}(d\varepsilon_{ij}^e + d\varepsilon_{ij}^p) = dW^e + dW^p \tag{C.35b}$$

The first part of the work is recovered upon unloading and the second part, dues to the irreversible plastic deformation, is not recovered. This latter part is the plastic work per unit volume,

$$dW^p = \sigma_{ij} d\varepsilon_{ij}^p \tag{C.35c}$$

Taking into account (C.23), the last relation can be expressed in terms of the principal deviatoric stresses,

$$dW^p = s_1 d\varepsilon_1^p + s_2 d\varepsilon_2^p + s_3 d\varepsilon_3^p \,. \tag{C.35c}$$

Without going into the details, the plastic work increment can be expressed in terms of the equivalent stress and equivalent plastic strain increment,

$$dW^p = \sigma_e d\varepsilon_p \quad . \tag{C.35c}$$

Accordingly, the Prandtl-Reuss relations (C.33b) for a work hardening material are,

$$d\varepsilon_{ij}^p = \frac{3}{2}\frac{dW^p}{\sigma_e^2} s_{ij} \tag{C.36}$$

It is assumed further that the amount of hardening depends on the total plastic work and is independent of the deformation path.

Therefore, in the yield criterion (C.8), parameter *K* is a function of the total plastic work changes for a work hardening material,

$$f(\sigma_{ij}) = K(W^p) \tag{C.37}$$

where,

$$W^p = \int \sigma_{ij} d\varepsilon_{ij}^p \tag{C.38}$$

When then V. Mises or equivalent stress is the yield function we obtain,

$$\sigma_e = K(W^p)\,. \tag{C.39}$$

Relationship (C.39) is obtained experimentally and used in (C.36) to obtain the plastic strain increments.

Strain hardening: The second approach to model hardening is to use the following equivalent plastic strain ε_p,

$$\varepsilon_p = \int d\varepsilon_{ij}^p\,. \tag{C.40}$$

Thus, (C.37) is replaced with,

$$f(\sigma_{ij}) = H(\varepsilon_p) \tag{C.41}$$

and for the V. Mises criterion, we have,

$$\sigma_e = H(\varepsilon_p)\,. \tag{C.42}$$

Relationship (C.42) is established with experimental measurements from the material in question and the plastic strain increments are computed from (C.33).

It can be shown that for the V. Mises criterion, both measurements of hardening (work and strain-hardening) are equivalent.

General approach to plastic stress-strain relations

In the previous sections, some aspects of incremental plasticity, based on (C.22), are outlined. It is also shown that the theory implies the V. Misses yield function. In this section we briefly present a more general formalism to determine stress-strain relations in the plastic range for any yield criterion.

This general approach is due to Drucker. We assume a strain hardening material and expresses, the yield function (C.8) differently for the sake of clarity. Thus, we assume that yielding occurs when,

$$F\left(\sigma_{ij}, \varepsilon_{ij}^{p}\right) = 0 . \tag{C.43}$$

It is important to keep in mind that unique relations do not exist between stress and strain in the plastic range; the strain depends not only on the final state of stress, but also on the loading history. Therefore, in developing theories of plasticity, the stress-strain relations of elasticity must be replaced by relations between increments of stress and strain (*incremental or flow theory of plasticity*).

Associated flow rule. The associated flow rule of classical plasticity is based on the Drucker's postulate. Consider an element of a strain-hardening material with an initial state of stress σ_{ij}^{0}. If an additional stress increment $d\sigma_{ij}$ is slowly applied on it and then removed, it is postulated that:

1. during loading, the additional stresses do positive work,
2. during the complete cycle of additional loading and unloading the additional stresses do positive work if plastic strains are produced. For a strain-hardening material, the work is zero only when the changes are purely elastic.

From the foregoing it follows that,

$$d\sigma_{ij} d\varepsilon_{ij}^{p} \geq 0 \tag{C.44a}$$

and,

$$\left(\sigma_{ij}-\sigma_{ij}^{0}\right)d\varepsilon_{ij}^{p}\geq 0\,. \tag{C.44b}$$

Because of (C.44b), the inequality $F \leq 0$ defines a convex region in stress space and the plastic strain increment vector $d\varepsilon_{ij}^{p}$ is normal to the loading surface $F = 0$. Accordingly, the incremental flow rule is,

$$d\varepsilon_{ij}^{p}=d\lambda\frac{\partial F}{\partial\sigma_{ij}},\quad d\lambda\geq 0\,. \tag{C.45a}$$

When F is the yield function we call (C.45a) the *associated flow rule*. For several material and especially metals, an associated flow rule works very well. Note that $\partial F/\partial\sigma_{ij}$ is the gradient of F and thus, the plastic strain increment vector is normal to the yield surface.

If we use another function we have the *non-associated flow rule*. Here a function $g\left(\sigma_{ij},\varepsilon_{ij}^{p}\right)$ is defined so that,

$$d\varepsilon_{ij}^{p}=d\lambda\frac{\partial g}{\partial\sigma_{ij}},\quad d\lambda\geq 0\,. \tag{C.45b}$$

Non-associated flow rule does not satisfy Drucker's postulate and have limited applications.

After yielding, subsequent loading surfaces pas through the stress point, representing the stress state in the stress space and (C.43) is always true. Thus, during loading we have,

$$dF=\frac{\partial F}{\partial\sigma_{ij}}d\sigma_{ij}+\frac{\partial F}{\partial\varepsilon_{mn}^{p}}d\varepsilon_{mn}^{p}=0\,. \tag{C.46}$$

This last relation is the so-called *consistency condition*. Next, we introduce (C.45a) in (C.46) and solve for $d\lambda$,

$$d\lambda=-\frac{\left(\partial F/\partial\sigma_{ij}\right)d\sigma_{ij}}{\left(\partial F/\partial\varepsilon_{mn}^{p}\right)\left(\partial F/\partial\sigma_{mn}\right)}=-\frac{\left(\partial F/\partial\sigma_{kl}\right)d\sigma_{kl}}{\left(\partial F/\partial\varepsilon_{mn}^{p}\right)\left(\partial F/\partial\sigma_{mn}\right)} \tag{C.47}$$

Introducing (C.47) in (C.45a) we obtain following complete increment plastic-strain stress relations,

$$d\varepsilon_{ij}^{p} = -\frac{\left(\partial F / \partial \sigma_{ij}\right)\left(\partial F / \partial \sigma_{kl}\right)}{\left(\partial F / \partial \varepsilon_{mn}^{p}\right)\left(\partial F / \partial \sigma_{mn}\right)} d\sigma_{kl} \tag{C.48}$$

The last equation implies that the plastic strain increments are proportional to the stress increments and (C.48) can be written as,

$$d\varepsilon_{ij}^{p} = H_{ijkl} d\sigma_{kl} \tag{C.49a}$$

with,

$$H_{ijkl} = -\frac{\left(\partial F / \partial \sigma_{ij}\right)\left(\partial F / \partial \sigma_{kl}\right)}{\left(\partial F / \partial \varepsilon_{mn}^{p}\right)\left(\partial F / \partial \sigma_{mn}\right)} \tag{C.49b}$$

Hardening rule. It was stated earlier that if loading is continued after the yield point and the material strain-hardens, the loading surface can changes its size, shape, and location in the stress space. The rule which accounts for such a modification of the loading surface during plastic flow is called hardening rule.

Isotropic hardening. The most widely used hardening rule assumes a uniform expansion of the initial yield surface as discussed earlier and shown in Fig. 5Ca. It is called isotropic hardening, because it assumes negligible anisotropy induced from plastic strains. This is the simplest possible hypothesis and widely used in classical plasticity for certain materials. The loading function depends on a single parameter H and may be written as,

$$F\left(\sigma_{ij}, H\right) = f\left(J_2, J_3\right) - H\left(\sigma_{ij}, \varepsilon_{ij}^{p}\right) \leq 0 . \tag{C.50}$$

Here $F\left(J_2, J_3\right)$ is a function of stresses only (i.e. loading function) and $H\left(\sigma_{ij}, \varepsilon_{ij}^{p}\right)$ is the hardening function. It is clear that for the V. Mises criterion, we have,

$$F\left(\sigma_{ij}, H\right) = \frac{1}{2} s_{ij} s_{ij} - H\left(\sigma_{ij}, \varepsilon_{ij}^{p}\right) \leq 0 . \tag{C.51}$$

As stated earlier there are two propositions for the function $H\left(\sigma_{ij}, \varepsilon_{ij}^{p}\right)$. The first one is based on the plastic work and the other on the plastic strains accumulated in the material, i.e.

$$W^{p} = \int \sigma_{ij} d\varepsilon_{ij}^{p} \tag{C.52}$$

and,

$$H\left(\varepsilon_{ij}^{p}\right)=\int\left[\frac{2}{3}d\varepsilon_{ij}^{p}d\varepsilon_{ij}^{p}\right]^{1/2}. \tag{C.53}$$

Equation (C.52), represents the plastic work done during deformation and (C.53) denotes the so-called "effective" or "equivalent" strain. Both integrals are taken over the strain path from some initial state. (Note that, (C.23) is implied in the plastic strain increments). The equivalent strain increment in (C.53) integrated over the strain path, provides a good measure of plastic distortions. When the hardening theory is formulated in terms of (C.52) the material is said to be work-hardenig. When the hardening theory is formulated in terms of (C.53) the material is said to be strain-hardening. As mentioned earlier, for the V. Mises plasticity the formulations of work-hardening and strain-hardening are equivalent. However, in general they give different result.

If F and H are both non-decreasing functions of their arguments, the loading surface will expand in the process of plastic deformation. On the other hand, if F and H are both decreasing functions, the material is said to be *strain-softening*. In this case the loading surface decreases in size.

Example 5: Show that

$$\frac{\partial J_2}{\partial \sigma_{pq}}=s_{pq} \qquad \frac{\partial\left(s_{mn}s_{mn}\right)^{1/2}}{\partial\sigma_{ij}}=\frac{s_{ij}}{\left(s_{mn}s_{mn}\right)^{1/2}}, \quad s_{mn}s_{mn}>0$$

where $J_2=\frac{1}{2}s_{ij}s_{ij}$ is the second invariant of the deviatoric stress tensor with components s_{ij}.

Solution

$$\frac{\partial J_2}{\partial\sigma_{pq}}=\frac{1}{2}\left(\frac{\partial s_{ij}}{\partial\sigma_{pq}}s_{ij}+s_{ij}\frac{\partial s_{ij}}{\partial\sigma_{pq}}\right)=\frac{\partial s_{ij}}{\partial\sigma_{pq}}s_{ij}=\frac{\partial}{\partial\sigma_{pq}}\left(\sigma_{ij}-\frac{1}{3}\delta_{ij}\sigma_{mm}\right)s_{ij}$$

$$=\left(\delta_{ip}\delta_{jq}-\frac{1}{3}\delta_{ij}\delta_{mp}\delta_{mq}\right)s_{ij}=s_{ij}\delta_{ip}\delta_{jq}-\frac{1}{3}s_{ij}\delta_{ij}\delta_{mp}\delta_{mq}=s_{pq}-\frac{1}{3}s_{ii}\delta_{pq}$$

$$\Rightarrow\frac{\partial J_2}{\partial\sigma_{pq}}=s_{pq}.$$

$$\frac{\partial\left(s_{mn}s_{mn}\right)^{1/2}}{\partial\sigma_{ij}}=\frac{1}{2}\frac{1}{\left(s_{mn}s_{mn}\right)^{1/2}}\left(\frac{\partial s_{mn}}{\partial\sigma_{ij}}s_{mn}+s_{mn}\frac{\partial s_{mn}}{\partial\sigma_{ij}}\right)=\frac{1}{\left(s_{mn}s_{mn}\right)^{1/2}}\left(\frac{\partial s_{mn}}{\partial\sigma_{ij}}s_{mn}\right)$$

$$=\frac{\delta_{mi}\delta_{nj}s_{mn}}{\left(s_{mn}s_{mn}\right)^{1/2}}=\frac{s_{ij}}{\left(s_{mn}s_{mn}\right)^{1/2}}.$$

Example 6: The associated flow rule in plasticity is when the function F in (C.45a) is the yield function. Show that if F is the V. Mises yield function, the associated flow rule gives the Prandtl - Reuss equations for the incremental plastic strains.

Solution

Consider the function (C.51) for a strain hardening material (C.53),

$$F\left(\sigma_{ij},H\right)=\frac{1}{2}s_{ij}s_{ij}-H\left(\varepsilon_{ij}^{p}\right)$$

Taking into account the results of the last example the flow rule (C.45a) becomes,

$$d\varepsilon_{ij}^{p}=d\lambda\frac{\partial F}{\partial\sigma_{ij}}=d\lambda\frac{1}{2}\frac{\partial(s_{pq}s_{pq})}{\partial\sigma_{ij}}=s_{ij}d\lambda$$

which is (C.22b). Thus, the Prandtl - Reuss equations imply the V Misses yield criterion.

Deformation theories of plasticity

So far we discussed a well-known incremental or flow theory of plasticity which relates the plastic strain increments to the current state of stress (C.33) by,

$$d\varepsilon_{ij}^{p}=\frac{3}{2}\frac{d\varepsilon_{p}}{\sigma_{e}}s_{ij} \qquad \text{(C.33bis)}$$

To obtain the total plastic strains, we need to integrate these equations over the entire loading history.

Hencky in 1924, proposed relations between total plastic strains and current stress state. Thus, instead of (C.33), we have,

$$\varepsilon_{ij}^{p}=\frac{3}{2}\frac{\varepsilon_{p}}{\sigma_{e}}s_{ij} \qquad \text{(C.54)}$$

In this theory, the strains are functions of the current stress state and independent of the loading history. Theories that model total plastic strains in this manner, are called *total* or *deformation theories* as opposed to the *incremental* or *flow theories of plasticity*.

It is evident that *deformation theories* are simpler than the *incremental theories.* However, they do not account for the load-path dependency of plastic strains. Thus, when the stresses are not increased continuously, (C.54) do to give accurate results. This situation appears in unloading-reloading: assume that after yielding, a specimen is unloaded, partially or completely and reloaded to a state of stress that does not produce yielding. In such a case, the plastic strains do not change but (C.54) indicate different values of plastic strains because the stresses have changed. This is not realistic because the plastic strains have not changed.

Nevertheless, the two types of theories give the same results for the case of proportional loading: $\sigma_{ij} = \kappa\sigma_{ij}^0$, where σ_{ij}^0 is a reference stress state and κ a constant. That is, if all stresses are increased proportionally, the incremental theory, reduces to the deformation theory. This is not difficult to verify because for $s_{ij} = \kappa s_{ij}^0$, $\sigma_e = \kappa\sigma_e^0$, (C.33) results in,

$$d\varepsilon_{ij}^p = \frac{3}{2}\frac{d\varepsilon_p}{\sigma_e^0}s_{ij}^0 \qquad \text{(C.55a)}$$

which upon integration gives,

$$\varepsilon_{ij}^p = \frac{3}{2}\frac{s_{ij}^0}{\sigma_e^0}\varepsilon_p = \frac{3}{2}\frac{s_{ij}}{\sigma_e}\varepsilon_p \qquad \text{(C.55b)}$$

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